NUMERICAL ANALYSIS OF THE RIEMANN ZETA FUNCTION; AND ITS CONCLUSIONS

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Keywords: Riemann Zeta, Riemann hypothesis, complex numbers, mathematical analysis

1. ABSTRACT

In our paper we report about the analysis of the *Riemann Zeta* function, based on the numeric values calculated at 10 million points of the $|Re(s)| < 1$ region of the complex plane, and the conclusions arising from the analysis. We also consider the studies related to the $E(s)$ function, introduced by us, to be a key result. We used the MathWorks software for the analyses.

2. RIEMANN ZETA FUNCTION

The Riemann Zeta function $\zeta(s)$ is a function of complex variable *s* that analytically continues the sum of the infinit series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s = a + bi \tag{1}$$

which converges when *a>1*. It is easy to see that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

is convergent for *a*>0, and that it is equal to $(1 - \frac{1}{2^s})\zeta(s)$ for *a>0*. Riemann showed that $\zeta(s)$ has an analytic extension to a meromorphic function on $\mathbb{C}$ having a single simple pole at *s*=1. Moreover, he proved the functional equation

$$\zeta(s) := \frac{1}{2} s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \zeta(1-s) \tag{2}$$

where $\Gamma(s)$ is the Gamma function. $\zeta(s) = 0$ at *s=-2,-4,-6,...* and $\zeta(s) \neq 0$ if *a>1.* The assertion that $\zeta(s) \neq 0$ on the vertical line is equivalent to the prime number theorem, asserting that $\frac{\pi(x)}{li(x)} \to 0$ as $x \to \infty$, where $\pi(x)$ is equal to number of prime numbers up to x, and $li(x) = \int_2^x \frac{du}{log u}$.

All the other (so called non trivial) zeros are located on the *critical* strip 0<a<1. Riemann asserted that the number of that zeros whose imaginary parts are between 0 and T>0 is approximately equal to

$$\frac{T}{2\pi} log \frac{T}{2\pi} - \frac{T}{2\pi}$$

with the error term $O(\frac{1}{T})$.

Riemann stated that all roots of the zeta-function lie on the critical line *a=½*, which is the famous Riemann Hypothesis. So far, there is no proof of the *Riemann hypothesis* that states that the real part of every non-trivial zero of the ζ-function is ½, thus the non-trivial zeros lie on the so called *critical line* consisting of complex numbers 1/2 + *it.*

Zeta function can be written in trigonometric form as follows:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^a}\left(\cos\left(-b log_e(n)\right) + \sin\left(-b log_e(n)\right) i\right) \tag{3}$$

In this paper, we also rely on the *Riemann-Siegel* function.

Riemann devised a formula for the calculation of the zeros on the critical line, but the formula was not published, and it was only somewhat later, in 1930, that Carl Siegel rediscovered it. Thus, the function is now called the *Riemann-Siegel formula*:

$$\zeta(s) \approx \sum_{n=1}^{\left\lfloor\sqrt{\frac{b}{2\pi}}\right\rfloor} \frac{1}{n^a}\cos(\theta(b) - b\log_e(n)) \qquad (4)$$

where:

$$\theta(b) = -\frac{b}{2}\log_e\left(\frac{2\pi}{b}\right) - \frac{b}{2} - \frac{\pi}{8} + \frac{1}{48b} + \frac{7}{5760b^3} + \frac{31}{80640b^5} + \cdots \qquad (5)$$

Between the Riemann-Siegel formula and the Zeta function it is true that the absolute values of all of the complex numbers with real part ½ are equal.

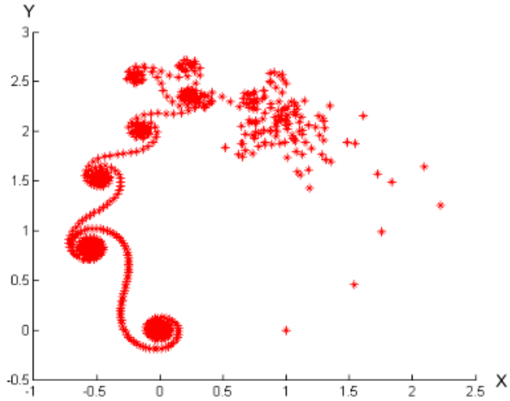

Figure 1.
The graphics of ζ_N

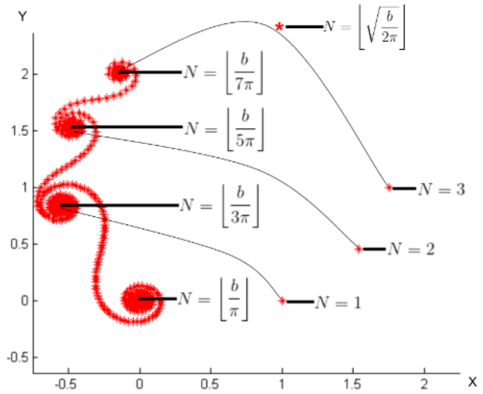

Figure 2.

3. THE EXAMINATION OF THE ZETA FUNCTION

Let's denote the *N*th partial sum with $\zeta_N(s)$ and plot them in the complex plane. Analysing the results, we see that increasing *N* results a more and more regular spiral diagram, as shown in Figure 1. The points of $\zeta_N(s)$ do not change their rotational direction until the difference $b\log_e(n) - b\log_e(n-1)$ exceeds one of the multiples of π. Given the fact that this difference decreases with increasing N, the points of the spiral are concentrating. We have found that by increasing the b, the value of $b\log_e(N) - b\log_e(N-1)$ at $N = \left\lfloor\frac{b}{k\pi}\right\rfloor$ becomes equal to $k\pi$.

Interesting result was obtained about the geometric relationship between partial sum at $N < \sqrt{\frac{b}{2\pi}}$ and $> \sqrt{\frac{b}{2\pi}}$; about the behaviour of the points of the spiral diagrams at a variety of real parts between 0 and 1 (Figure 2). Note that value $\sqrt{\frac{b}{2\pi}}$ plays a role in the case of the Riemann-Siegel function as well.

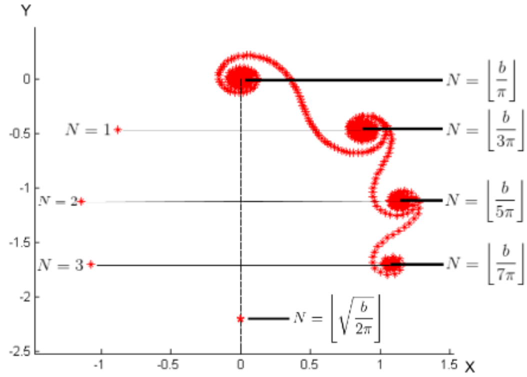

Figure 3.

3.1. The case of Re(s) = ½

In the case of $s = \frac{1}{2} + it$, it was found that partial sums at $N < \sqrt{\frac{b}{2\pi}}$ and $N > \sqrt{\frac{b}{2\pi}}$ are mirror images of each other, projections to an "imaginary" line at the points of partial sums $N = k$ and $N = \left\lfloor\frac{b}{(2+k)\pi}\right\rfloor$ are in a mirror image relationship with each other (Figure 3). Our calculations indicate that exists b where $\zeta_N(\frac{1}{2} + bi)$=0. (We will proof that $\zeta_N(s)$=0 cannot be true if $\neq \frac{1}{2}$.)

3.2. The case of $Re(s) \neq ½$

Our calculations show that for any arbitrary imaginary part value is true that in the case of $Re(s)=\frac{1}{2}$, the “last” spiral section between $\left\lfloor\frac{b}{3\pi}\right\rfloor$ and $\left\lfloor\frac{b}{\pi}\right\rfloor$ is very close to 1 (Figure 4). In the case of $Re(s)=\frac{1}{2}$, the graph of the partial sums takes the shape shown in Figure 5, that is, the lengths of the straight sections of the two branches, and the angle between them is the same. Because the change in the real part of the Zeta function does not affect the change of the angle between the vectors of the individual *Nth* and *N*-1st partial sums, only the distance, it remains that the two branches “assimilate” to each other, but the length of the sections between the partial sums are not equal (see Figure 5).

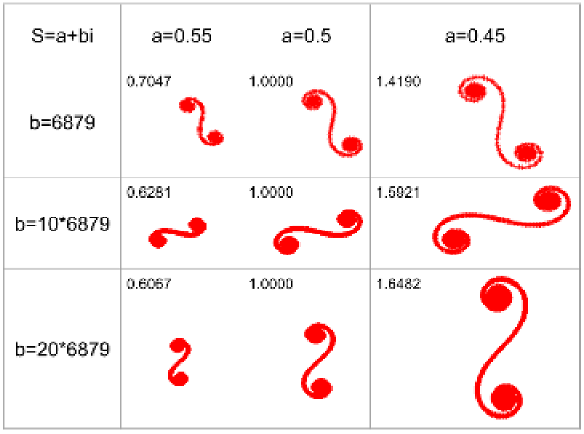

Figure 4.

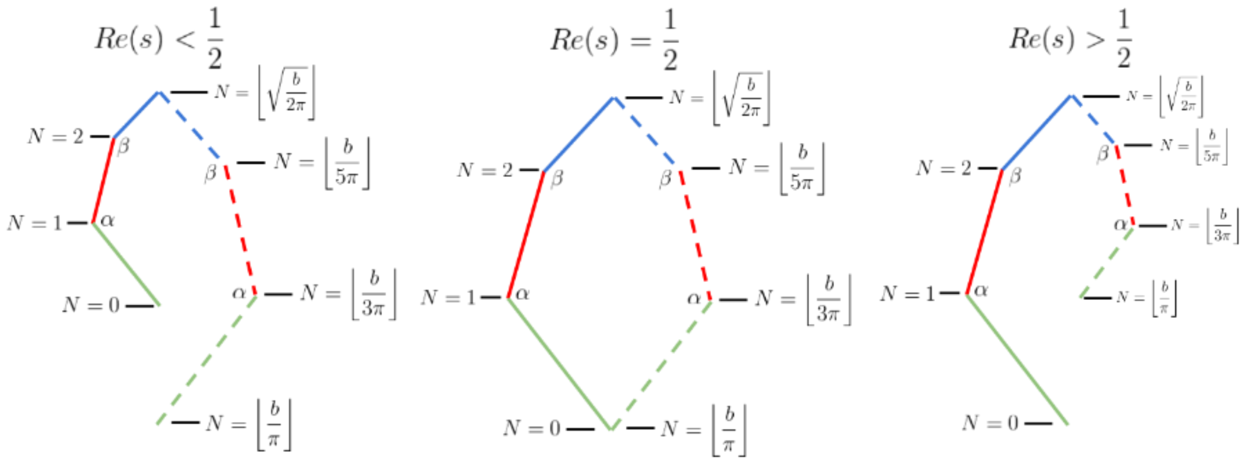

Figure 5.

The “last” spiral length can be calculated using the following expression:

$$\sum_{n=\left\lfloor\frac{b}{2\pi}\right\rfloor}^{\left\lfloor\frac{b}{2\pi}\right\rfloor+\left\lfloor\sqrt{\frac{b}{2\pi}}\right\rfloor} \frac{2}{\sqrt{2n^{a}}} \tag{6}$$

that can be approximated by the expression:

$$\frac{\left(\frac{b}{2\pi}\right)^{\frac{1}{2}-a}}{n^{1-a}} \tag{7}$$

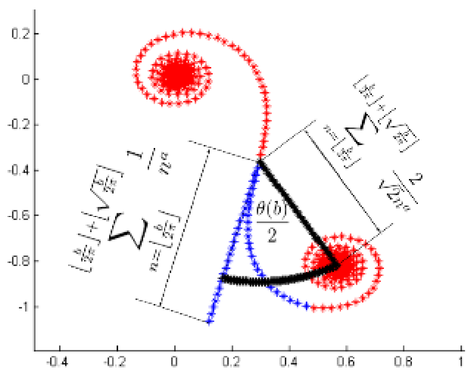

Figure 6.

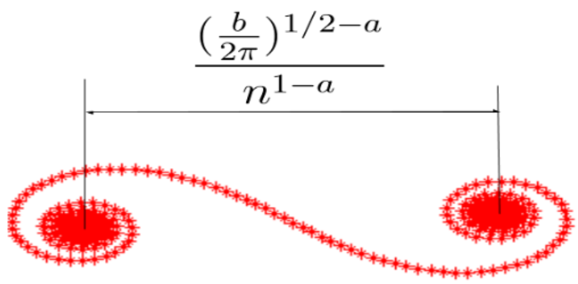

Figure 7.

From this observation we construct a new approximate function, which uses the Riemann-Siegel function as well.

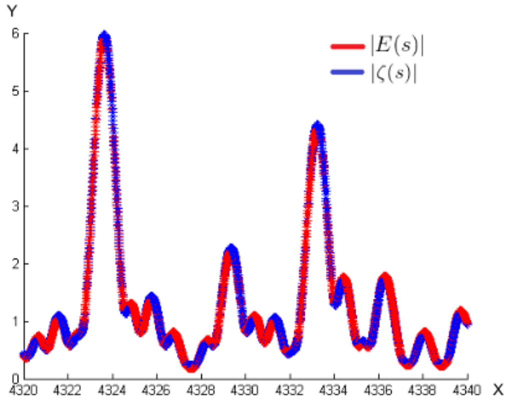

Figure 8.

4. NEW APPROXIMATE FUNCTION

Based on the numerical experience that the ζ_N partial sums are disposed symmetrically starting from $N = \sqrt{\frac{b}{2\pi}}$, we prepared the following approximate function:

$$E(s) = \sum_{n=1}^{\left\lfloor\sqrt{\frac{b}{2\pi}}\right\rfloor} \frac{1}{n^a}\cos\big(\theta(b) - b\log_e(n)\big) + \frac{1}{n^a}\sin\big(\theta(b) - b\log_e(n)\big)\, i + \frac{(\frac{b}{2\pi})^{\frac{1}{2}-a}}{n^{1-a}}\cos\big(\theta(b) - b\log_e(n)\big) - \frac{(\frac{b}{2\pi})^{\frac{1}{2}-a}}{n^{1-a}}\sin\big(\theta(b) - b\log_e(n)\big) i \tag{8}$$

Based on the calculated values of several million points, it can be stated that:
$|\zeta(s)| \approx |E(s)|$ (See Figure 8 as an illustration)

The difference of $|\zeta(s) - E(s)|$ is shown in Figure 9.

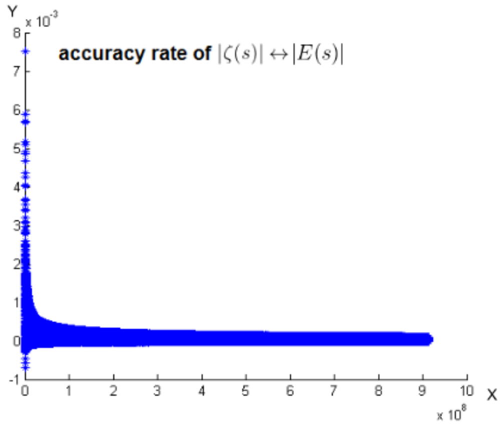

Figure 9.

The function $E(s)$ can be written in the form of $F_1(s) + F_2(s)i$, where:

$$F_1(s) = \sum_{n=1}^{\left\lfloor\sqrt{\frac{b}{2\pi}}\right\rfloor} \frac{1}{n^a}\cos\big(\theta(b) - b\log_e(n)\big) + \frac{(\frac{b}{2\pi})^{\frac{1}{2}-a}}{n^{1-a}}\cos\big(\theta(b) - b\log_e(n)\big) \tag{9}$$

$$F_2(s) = \sum_{n=1}^{\left\lfloor\sqrt{\frac{b}{2\pi}}\right\rfloor} \frac{1}{n^a}\sin\big(\theta(b) - b\log_e(n)\big) - \frac{(\frac{b}{2\pi})^{\frac{1}{2}-a}}{n^{1-a}}\sin\big(\theta(b) - b\log_e(n)\big) \tag{10}$$

The graphs shown in the Figures 10 and 11 serve as illustrations of the numerical analysis of $F_1(s)$ and $F_2(s)$. In these we can see that zero points of $F_1(s)$ and $F_2(s)$ do not overlap, thus $E(s)$ has no root in the case of any $Re(s) \neq \frac{1}{2}$.

Because of the $|\zeta_N(s)| \approx |E_N(s)|$ relationship this also means that $\zeta_N(s)$ has no root, which seems to prove the *Riemann hypothesis*.

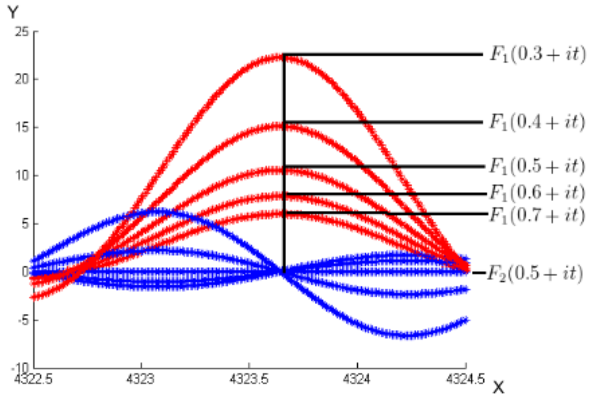

Figure 10.

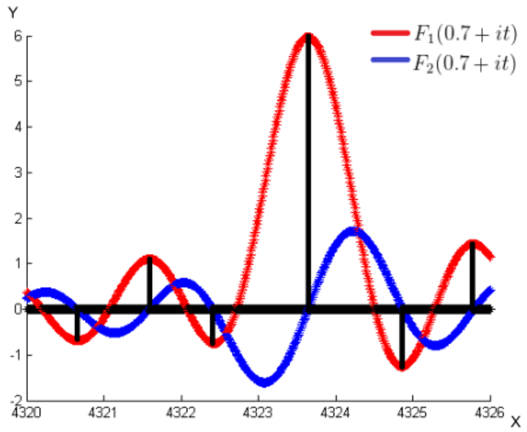

Figure 11.

The graph in Figure 10 also shows that where the $F_2(s)$ disappears in the cases of $Re(s) \neq \frac{1}{2}$ there $F_1(s) \neq 0$ (it has a maximum here), and that proves the *Riemann hypothesis* for the cases calculated by us.

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