NEW OPERATIONS IN THREE-VALUED LOGIC

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1. SUMMARY

Three-valued logic and its application are currently in the focus of many researches. The main challenges of the simple application is to solve some problems connected to the traditional three valued logic [1], and the difficulties with arithmetic applications [2]. In this paper two new operation groups are introduced to relieve these problems. The application of these operations is presented via some examples.

2. BACKGROUND AND RELEVANCE OF THE RESEARCH

2.1 The background of three-valued logic

"Lukasiewitz raised first the possibility of three-valued logic in which a statement can have three values: true, false and possible (e.g. indefinable).This theory has practical significance. John von Neumann was one of the discoverers of the applicability in quantum mechanics." [9] In the 1950s researchers had carried out experiments on computers with ternary systems in the Soviet Union, but the development of binary technology closed this direction for many years. By now, the circuit implementation of the traditional operations of three-valued logic has already been solved. In this field Hungary also achieved significant results in the 1970s. [3]

2.2 The current directions of the three-valued logic research

Presently, three-valued logics and the necessity of their applications have come back to the forefront of scientific interest.[1]

To take some examples: SQL has three-valued logic (cf. Rubinson, 2007) [6], since NULL is unknown (as it indicates that a data value does not exist in the database), while in the classic two-valued logic, every values are known. Avron and Konikowska (2008) [4] developed a three-valued logic to describe Pawlak's[2] rough sets[3] whereby the value t corresponds to the positive region of a set, the value f corresponds to the negative region, and the undefined value u (which represents a non-classical value: unknown) corresponds to the border of the set. The three-valued logic is also used for parametric shape analysis. In Sagiv's, Reps' and Wilhelm's researches (2002) [7], two-valued and three-valued logical structures are used to represent concrete and abstract stores; and the predicate-update formulae are evaluated in three-valued logic. Three-valued logic and ternary systems are used in the field of biology, as well. Chastain et al. (2012) [5] implemented the immune control logics for their example networks using Kleene's[4] three-valued logic.

3. THE TRADITIONAL STRUCTURE OF THREE-VALUED LOGIC

The three logical values are true (T), false (F) and Possible (P). The last one has another names, too, e.g. John von Neumann named it Maybe (M) [8].

3.1 Three-valued logic expressed in numbers

Numbers can be associated to the logical values.

3.1.1 Three-valued logic in the ternary system

In the ternary system F can be assigned to 0, and similarly P to 1 and T to 2. This assignment is used in one of our examples showing the truth table of half adder in ternary system.

3.1.2 More numerical assignments

There are other assignments, too; such as -1 is assigned to F (like 0 to P and 1 to T). This is well suited for e.g. three-way controllers. There is also an assignment, in which the Boolean logic is partially retained, so that F is assigned to 0, P to 1/2 and T to 1. The latter is called Lukasiewicz's logic assignment. T is not only a single numeri-

[1] In 2012 (13-17 August), a workshop was organised about these topics in Poland. http://paulegre.free.fr/TrivalentESSLLI/index.html

[2] Zdzisław I. Pawlak (1923–2006) was a Polish computer scientist.

[3] A rough set is a formal approximation of a crisp set (i.e. conventional set) in terms of a pair of sets which give the lower and the upper approximation of the original set.

[4] Stephen Cole Kleene (1909–1994) was an American mathematician, one of the pioneers of theoretical computer science.

cal value, but also can be assigned to an interval. In this case, the three-valued logic approaches to the fuzzy logic.

3.2 Standard operations of the three-valued logic

The complement operation of traditional operations (disjunction, equivalence, implication, complement formation) of the three-valued logic is called negation in more curriculums [10]. According to Otto Szittya [2], the more specific term 'complement' is used in this paper.

3.2.1 Complement formation

A	¬A
F	T
P	P
T	F

Table 1.
The truth table of complement formation

Denote that the numerical assignment in which -1, 0, 1 are used, the sum of the complement and the original number is 0.

4. NEW OPERATIONS OF THE THREE-VALUED LOGIC

By using the original operations of three-valued logic several logical and arithmetical problems cannot be solved, therefore new operations are needed. Other proposals of the research also gave inspiration towards creating new operations. The John von Neumann's quantum logic and researches of the 21st century called for the consideration with the tools of logic the timing of events and the handling of the possibilities-becoming-certainties, together with the demand for algebraic tracking of the phenomena. One pair of new operations handle changing chances, the other pair handle possibilities becoming certainties.

4.1 Two groups of the three-valued operations

4.1.1 Chances operations

Let be the operation of increasing chances ↑, the operation of decreasing chances ↓ of which the truth table is (Table 2).

A	↑A	↓A
F	P	F
P	T	F
T	T	P

Table 2.
Truth table of chances operations

Some identity of these two kinds of chances operations in three-valued logic are

$$\uparrow(A\vee B) = \uparrow A \vee \uparrow B,\quad \uparrow(A\wedge B) = \uparrow A \wedge \uparrow B,$$
$$\downarrow(A\vee B) = \downarrow A \vee \downarrow B \text{ and } \downarrow(A\wedge B) = \downarrow A \wedge \downarrow B$$

$$\downarrow\neg A = \neg\uparrow A,\quad \neg\downarrow A = \uparrow\neg A,$$
$$\downarrow A = \downarrow\uparrow\downarrow A \text{ and } \uparrow A = \uparrow\downarrow\uparrow A$$

$$\downarrow\neg\uparrow\downarrow A = \neg\uparrow\neg\downarrow\uparrow A,\quad \neg\downarrow\neg\uparrow\downarrow A = \uparrow\neg\downarrow\uparrow A,$$
$$\neg\downarrow A = \uparrow\neg A = \uparrow\neg\uparrow\downarrow A \text{ and } \neg\uparrow A = \downarrow\neg A = \downarrow\neg\downarrow\uparrow A.$$

The following not-satisfied expressions are interesting, too:

$$\uparrow\downarrow A \neq \downarrow\uparrow A,\quad \neg\downarrow A \neq \downarrow\neg A \text{ and } \neg\uparrow A \neq \uparrow\neg A.$$

4.1.2 Certainties operations

Let denote the certainties operations as ⇑ and ⇓ of which the truth table is (Table 3).

A	**⇑A**	**⇓A**	**⇑A⇔⇓A**	**⇑↓A**	**⇓↑A**
F	F	F	T	F	F
P	T	F	F	F	T
T	T	T	T	T	T

Table 3.
Truth table of certainties operations

The notation of equivalence is ⇔ in Table 3. It can be seen that one of the certainties operations is sufficient, e.g. ⇓.

Some important identities of two ***certainties operations*** are

$$\Uparrow(A\vee B) = \Uparrow A \vee \Uparrow B,\quad \Uparrow(A\wedge B) = \Uparrow A \wedge \Uparrow B,$$
$$\Downarrow(A\vee B) = \Downarrow A \vee \Downarrow B \text{ and } \Downarrow(A\wedge B) = \Downarrow A \wedge \Downarrow B.$$

4.2 Samples for using of the new operations

Three simple examples are shown to prove the practical significance of the new operations. One is arithmetic, the second is logical, and the third is a theoretical solution of a control engineering problem.

4.2.1 Half-adder in the three-valued logic

Here is described a theoretical example of half-adder based on three-valued logic: A and B are two addend, C is transmission to higher local value, and S is the sum of the digits of the original local value (Table 4).

A	B	C	S	Decimal
0	0	0	0	0
0	1	0	1	1
0	2	0	2	2
1	0	0	1	1
1	1	0	2	2
1	2	1	0	3
2	0	0	2	2
2	1	1	0	3
2	2	1	1	4

Table 4.
The half adder's arithmetic table

The formula of logical function of transfer is

$$C=\downarrow A \wedge B \vee A \wedge \downarrow B \qquad (1)$$

A	B	$\downarrow A \wedge B$	$A \wedge \downarrow B$	$C=\downarrow A \wedge B \vee A \wedge \downarrow B$
0	0	0	0	0
0	1	0	0	0
0	2	0	0	0
1	0	0	0	0
1	1	0	0	0
1	2	0	1	1
2	0	0	0	0
2	1	1	0	1
2	2	1	1	1

Table 5.
Truth table of transfer express with number

The formula of logical function of the result is (Table 6):

$$S=\downarrow A \wedge \downarrow B \vee {}^{\Downarrow}(\downarrow A \Leftrightarrow \downarrow B \wedge \uparrow A \Leftrightarrow \uparrow B) \vee (A \vee B) \wedge {}^{\Downarrow}\neg C \qquad (2)$$

A	B	$E=\downarrow A \wedge \downarrow B$	$F={\Downarrow}(\downarrow A \Leftrightarrow \downarrow B \wedge \uparrow A \Leftrightarrow \uparrow B)$	$G=(A \vee B) \wedge {\Downarrow}\neg C$	$S=E \vee F \vee G$
0	0	0	0	0	0
0	1	0	0	1	1
0	2	0	0	2	2
1	0	0	0	1	1
1	1	0	2	1	2
1	2	0	0	0	0
2	0	0	0	2	2
2	1	0	0	0	0
2	2	1	0	0	1

Table 6.
Truth table of the result express with number

4.2.2 Walking in the rain

A very simple example is shown for solving an educational inference practice task in artificial intelligence with three-valued logic. Three declarative sentences are given:

I might go for a walk when it is not raining.

I do not walk alone in the rain.

My girlfriend loves walking in the rain, and her wish is my command.

Let be note B: I am with my girlfriend; E: It is raining; S: I am walking.

The transfer function is (Table 7):

$$S=\downarrow \neg E \vee B \wedge E \qquad (3)$$

B	E	$\neg E$	$\downarrow \neg E$	$B \wedge E$	$S=\downarrow \neg E \vee B \wedge E$
F	F	T	P	F	P
F	T	F	F	F	F
T	F	T	P	F	P
T	T	F	F	T	T
T	P	P	F	P	P
P	F	T	P	F	P
P	T	F	F	P	P
P	P	P	F	P	P
F	P	P	F	F	F

Table 7.
Truth table of logic of walking in the rain

On top of Table 7 there are inputs without possible (P). The first sentence is inherently ambiguous; the third one does not exclude the possibility that his girlfriend would take a walk in dry weather, but this is not certain. Therefore, P must appear in the outputs of the first four pairing. The outputs of the other five rows are logical.

4.2.3 Vehicle control with tree-valued logic

We would like to run a self-handling vehicle on pre-built rails. The vehicle has three gears: fast (T), medium (P) and slow (F). Gear can be changed with only one stage at a time for safe operation. The track mostly consists of straight lines but at some points 90-degree bends are incorporated. The bends must be approached with slow speed (geared from medium speed) by the vehicle. After the bend, the speed must be changed gradually back to fast speed. The vehicle can go with maximum speed on the straight sections. The curves are indicated by tables standing on the end of the previous straight line. The vehicle is equipped with a sensor that is able to detect the table from a given distance. The possible results of the detection (E) are the following: the sensor cannot detect anything (F), it detects something (P), or the sensor recognizes a table (T). This detection occurs at certain time intervals. The new speed must be determined according to the values of detection and the current (original) speed (RS) by the controller.

Table 8 shows the three-valued logic values as a function of detection and the original speed. The last column shows the logical values of the new gear (V) of the vehicle. The control of the vehicle with regard to the criteria above (with one exception) can be achieved with the following simple rule:

$$V=\neg E \wedge\!\uparrow RS. \qquad (4)$$

Detection *E*	¬*E*	Original speed (*RS*)	New speed (¬*E* ∧↑*RS*)
F	T	T	T
F	T	P	T
F	T	F	P
P	P	T	P
P	P	P	P
P	P	F	P
T	F	T	P
T	F	P	F
T	F	F	F

Table 8.
Truth table of the logic of Vehicle Control

The grey line does not fit the rule, but it cannot happen in practice, because of the sensor always detects something (P) before the recognition (T). The grid line shows small speed drifts, which helps to approach the turn with a low, medium speed.

5. CONCLUSIONS

- Nowadays there is a renaissance in the research and application of three-valued logic.
- Introducing the chances and certainties operations simplifies the solution of problems presented difficulties earlier, and these operations have got many interesting mathematical properties.
- Especially, Formula (3) and Formula (4) show that the introduction of two chances operations (↓ and ↑) simplifies the solution of problems of conclusion and vehicle control.
- The new operations can be programmed easily, so the circuit realization can be well demonstrated in the education.

6. REFERENCES

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